REMARKS ON THE PAPER BY V.S. GUBENKO, "SOME CONTACT PROBLEMS OF THE THEORY OF ELASTICITY AND FRACTIONAL DIFFERENTIATION"

(ZAMECHANIIA K RABOTE V.C. GUBENKO, "NEKOTORYE ZADACHI TEORII UPRUGOSTI I DROBNOYE DIFFERENTSIROVANYE")

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In the paper mentioned, an attempt has been made to reduce the most simple, axially-symmetric, mixed problem of the theory of the Newtonian potential for the half-space

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \qquad z \geqslant 0 \tag{1}$$

$$u\Big|_{z=0} = F(r), \qquad r < a, \qquad\qquad \frac{\partial u}{\partial z}\Big|_{z=0} = 0, \qquad r > a \tag{2}$$

to a certain problem of the logarithmic potential for the half-plane. Differentiating n times with respect to r^2 the left-hand side of equation (1), and then formally putting $n = -1/2$, the author obtained the equation

$$\frac{\partial^2 u_{-1/2}}{\partial r^2} + \frac{\partial^2 u_{-1/2}}{\partial z^2} = 0 \tag{2*}$$

Here and in the following, author's equations are given with his numbers marked with asterisks; u_n denotes

$$c\left(\frac{\partial}{\partial (r^2)}\right)^n \quad (c \text{ — some constant, } c \neq 0)$$

The derivatives of the fractional order are introduced by the author in a formal-empirical way: in the expression for the derivative of order n of a power function, the numerical coefficient is represented as a ratio of Gamma-functions; the integer n in their arguments will in the following be replaced by an arbitrary number. In the case of functions represented by power series, it offers the possibility of obtaining - by differentiating term by term - the series respresenting the derivatives of an arbitrary, non-integer order.

There is no necessity, however, for such an indirect procedure.

Non-formal definition of the derivative of a negative order is given by the integral transformation [1]:

$$D^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-y)^{\nu-1} f(y)\, dy \qquad (\nu > 0)$$

This does not require $f(x)$ to be analytical, but only integrable (in the sense of Riemann and Stieltjes). Zero as the lower limit of the integral is not compulsory and may be replaced, if necessary, by any other number. The derivative of a positive order $\mu > 0$, $\mu = [\mu] + 1 - \nu$ is defined as $D^{[\mu]+1} D^{-\nu}$. Replacing the variable of differentiation by x^2, we obtain

$$D_{(x^2)}{}^{-\nu} f(x) = \frac{2}{\Gamma(\nu)} \int_0^x (x^2 - y^2)^{\nu-1}\, y f(y)\, dy \tag{3}$$

and in particular

$$D_{(x^2)}{}^{-1/2} f(x) = \frac{2}{\sqrt{\pi}} \int_0^x (x^2 - y^2)^{-1/2}\, y f(y)\, dy \tag{4}$$

$$D_{(x^2)}{}^{1/2} f(x) = \frac{1}{\sqrt{\pi}\, x} \frac{d}{dx} \int_0^x (x^2 - y^2)^{-1/2}\, y f(y)\, dy \tag{5}$$

From this point of view, the result expressed by equation (2*) does not appear to be new. Exactly by the transformation (4), Mossakovskii [2] reduces the axially-symmetric mixed problem of the theory of elasticity to the problem of linear conjugateness in the plane $x + iz$. In the particular case being considered, the result of Mossakovskii is obtained immediately, if the solution of equation (1) for the case of continuous circular area (on the boundary $z = 0$) is represented in the form

$$u(r, z) = \int_0^\infty e^{-zt} J_0(rt) f(t)\, dt \tag{6}$$

Thus with Sonine's integral

$$\int_0^\infty \frac{J_0(rt)\, r dr}{\sqrt{x^2 - r^2}} = \frac{\sin xt}{t}$$

and assuming the admissibility of altering the order of integration (which will be always assumed when necessary) we obtain

$$v(x, z) = \int_0^x \frac{u(r, z)\, r dr}{\sqrt{x^2 - r^2}} = \int_0^\infty e^{-zt} \sin xt \frac{f(t)}{t}\, dt \tag{7}$$

Hence, it is clear that $v(x, z)$ is a plane harmonic function, being odd with respect to x. Notation $v(x, z)$ corresponds to the author's notation $u_{-1/2}(r, z)$. The author obtains this result when proving equation

(2*) by differentiation, of order − 1/2 of integral (6) representing $u(r, z)$. But in the following, the author, using a superficial analogy, commits an error in assuming that the boundary conditions for $u_{-1/2}(r, z)$ are the same as for $u(r, z)$. Wishing to demonstrate his procedure with the example of obtaining the known expression for the pressure under a plane punch, $p = c/(a^2 - r^2)^{1/2}$, where

$$u(r, 0) = h \text{ (under the punch)}, \quad u_2'(r, 0) = 0 \text{ (outside the punch)} \qquad (8^*)$$

he writes the boundary conditions for $u_{-1/2}$ in the following form

$$u_{-1/2}(r, 0) = g\sqrt{r^2} \text{ (under the punch)}$$

$$\partial u_{-1/2}(r, 0)/\partial z = 0 \quad \text{(outside the punch)} \qquad (10^*)$$

It may be considered as a "misprint" $g\sqrt{r^2}$, instead of gr, as from the text it follows that the author has in mind gr and not $g\sqrt{r^2}$.

But the second of equations (10*) may not be considered as a "misprint", since the further discussion shows that it is the starting point for him. The author takes the known solution of the plane problem in the case (10*), and writes the equations [3, 4]

$$p_{-1/2} = \frac{p_0}{\pi\sqrt{a^2 - r^2}} - \frac{Eg\sqrt{r^2}}{2(1 - \nu^2)\sqrt{a^2 - r^2}} \qquad (11^*)$$

(Here a "misprint" again $\sqrt{r^2}$ instead of r). Having this, he calculates the pressure p (denoted in his paper also by p_0) under the circular punch, taking the derivative of order 1/2 by means of differentiation, term by term, of the power series expansion of the right-hand side of (11*). He claims that in this way the correct result is obtained. But in such a way it may not be obtained. In fact, by inversion of (7), or - which is the same - using (5), we have

$$u(r, z) = \frac{2}{\pi r}\frac{d}{dr}\int_0^r \frac{v(x, z)\,x\,dx}{\sqrt{r^2 - x^2}} \qquad (8)$$

and consequently

$$p(r) = \frac{2}{\pi r}\frac{d}{dr}\int_0^r \frac{p_{-1/2}(x)\,x\,dx}{\sqrt{r^2 - x^2}} \qquad (9)$$

Introducing (2) into this expression and performing the integration, we obtain

$$p(r) = \frac{2a}{\pi(a^2 - r^2)}\left[\frac{p_0}{\pi r} - \frac{Eg}{2(1 - \nu)}E\left(\frac{r}{a}\right)\right] \qquad (10)$$

Since the fractional differentiation term by term is equivalent to the determination of integral (9) by a series expansion, the author would have obtained the same result, if he had calculated correctly. But he attempts to justify his more than careless calculations and states that in the process of differentiation "the terms containing the singularities of order (– 1) and higher " should be neglected. Judging from reference [5], this concerns the singularity at $r = 0$, caused by the singularity of order (– 1/2) in the first term of the right-hand side of (11*). But the second term is not better, and it should be clear from the beginning that the differentiation increases the order of the singularity at point $r = a$ by – 1/2 and, consequently, if all the singularities of order (– 1) are neglected, zero remains on the right-hand side.

Although the author refers to paper [2], yet he does not know that transformation (7) does not lead to a mixed problem, but to the problem of Dirichlet for a plane harmonic function. In fact, from (6) we find

$$\frac{\partial u}{\partial z} = -\int_0^\infty e^{-zt} J_0(rt)\, t f(t)\, dt = -\frac{1}{r}\frac{d}{dr}\left\{r\int_0^\infty e^{-zt} J_1(rt) f(t)\, dt\right\} \tag{11}$$

Hence, on the basis of the second boundary condition

$$\frac{d}{dr}\left\{r\int_0^\infty J_1(rt) f(t)\, dt\right\} = 0 \qquad (r \quad a) \tag{12}$$

Consequently,

$$\int_0^\infty J_1(rt) f(t)\, dt = \frac{c}{r} \qquad (r > a) \tag{13}$$

Multiplying both sides by $1/(r^2 - x^2)^{-1/2}$ and integrating with respect to r from x to ∞ (with the condition $x > a$) and using the known integral

$$\int_x^\infty \frac{J_1(rt)\, dr}{\sqrt{r^2 - x^2}} = \frac{\sin xt}{xt}$$

we obtain the equation

$$\int_0^\infty \sin xt \, \frac{f(t)}{t}\, dt = \frac{\pi}{2} c \tag{14}$$

Thus, the boundary conditions for $v(x, z)$ have the form

$$v(x, 0) = \Phi(x) = \begin{cases} \displaystyle\int_0^x \frac{F(r)\, r\, dr}{\sqrt{x^2 - r^2}}, & |x| < a \\ \text{const}, & |x| > a \end{cases} \tag{15}$$

Taking now the known solution of the problem of Dirichlet

$$v(x, z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \Phi'(t) \operatorname{arc\,tg} \frac{x - t}{z}\, dt \tag{16}$$

and using transformation (8), we may obtain the closed expression for the solution of the axially-symmetric problem, containing complex variables and being the same as in paper [6] and book [7]. But this calculation is rather cumbersome. In the given simple example, the use of the results of the plane problem is not suitable, since the auxiliary function $f(t)$ may be found immediately by inversion of the Fourier integral determining the boundary condition

$$\int_0^\infty \sin xt \frac{f(t)}{t}\, dt = \Phi(x) \tag{17}$$

Hence

$$f(t) = \frac{2}{\pi} \int_0^\infty \Phi'(x) \cos tx\, dt \tag{18}$$

In consequence of (15), the inversion exists at any integrable $F(r)$. Introducing the latest expression into (6) and (11), and changing the order of integrations, we obtain for the kernels the following Hankel integrals

$$\int_0^\infty e^{-zt} \cos xt\, J_0(rt)\, dt = \operatorname{Re} \frac{1}{\sqrt{(z+ix)^2 + r^2}}$$

$$\int_0^\infty e^{-zt} \cos xt\, J_0(rt)\, dt = -\frac{1}{r} \operatorname{Re} \frac{z+ix}{\sqrt{(z+ix)^2 + r^2}}$$

Finally we have

$$u(r, z) = \frac{2}{\pi} \operatorname{Re} \int_0^a \frac{\Phi'(x)\, dx}{\sqrt{(z+ix)^2 + r^2}} \tag{19}$$

$$\frac{\partial u(r, z)}{\partial z} = \frac{2}{\pi r} \operatorname{Re} \int_0^a \frac{(z+ix)\, \Phi'(x)\, dx}{\sqrt{(z+ix)^2 + r^2}} \tag{20}$$

a is the upper limit, because $\Phi'(x) = 0$ at $x > a$. For $z = 0$ it is

$$u\big|_{z=0} = \frac{2}{\pi} \int_0^{\min(r,\, a)} \frac{\Phi'(x)\, dx}{\sqrt{r^2 - x^2}} \tag{21}$$

$$\frac{\partial u}{\partial z}\bigg|_{z=0} = \frac{2}{\pi r} \frac{d}{dr} \int_r^a \frac{\Phi'(x)\, x\, dx}{\sqrt{x^2 - r^2}}, \qquad r < a \tag{22}$$

Expressions (19) and (20) differ in notations only from those of book [7]. In paper [6], the corresponding expressions have some other form.

If the author established correctly the boundary conditions for $u_{-1/2}(r, z)$, on the basis of (15), he would have at $F(r) = 1$

$$u_{-1/2}\big|_{z=0} = \begin{cases} \min(r, a) & (r > 0) \\ \max(r, -a) & (r < 0) \end{cases}$$

From (16) we easily obtain

$$p_{-1/2}(r) = c\,\frac{\partial u_{-1/2}}{\partial z}\bigg|_{z=0} = c\,\frac{1}{\pi}\ln\left|\frac{a-r}{a+r}\right|$$

His expression (11*) should have this form. Thus, the fractional differentiation would obviously lead to the correct result. In fact, introducing the last expression into (9) and calculating the integral:

$$\frac{1}{\pi}\int_0^r \ln\left|\frac{a-x}{a+x}\right| \frac{x\,dx}{\sqrt{r^2-x^2}} = \sqrt{a^2-r^2} - a$$

we obtain

$$p(r) = \frac{A}{\sqrt{a^2-r^2}}$$

as it should be.

What was said for the circular punch saves us from the necessity of commenting on the expression for the pressure under the annular punch with a plane base, which was obtained in the same way.

BIBLIOGRAPHY

1. Ditkin, V.A. and Kuznetsov, P.I., *Spravochnik po operatsionnomu ischisleniiu (Handbook of Operational Calculus)*. GTTI, pp. 40-41, 1951.

2. Mossakovskii, V.I., Osnovnaia smeshannaia zadacha teorii uprugosti dlia poluprostranstva s krugovoi liniei razdela granichnykh uslovii (Basic mixed problem of the theory of elasticity for a half-space with circular line of separation of boundary conditions). *PMM* Vol. 18, No. 2, 1954.

3. Muskhelishvili, N.I., *Nekotorye osnovnye zadachi matematicheskoi teorii uprugosti (Some Basic Problems of the Mathematical Theory of Elasticity)*. 4th Ed., *Akad. Nauk SSSR*, p. 445, 1954.

4. Galin, L.A., *Kontaktnye zadachi teorii uprugosti (Contact Problems of the Theory of Plasticity)*. GTTI, p. 64, 1953.

5. Leonov, M.Ia., Obshchaia zadacha o davlenii krugovo shtampa na uprugoe poluprostranstva (General problem of the pressure of a circular punch on an elastic half-space).. *PMM* Vol. 17, No. 1, 1953.

6. Rostovtsev, N.A., Kompleksnye funktsii napriazhenii v osesimmetrichnoi kontaktnoi zadache teorii uprugosti (Complex stress functions in the axially-symmetric contact problem of the theory of elasticity). *PMM* Vol. 17, No. 5, 1953.

7. Green, A. and Zerna, W., *Theoretical Elasticity*. Oxford, p. 172, 1953.

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